Strong connexivity

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Connexive logics aim to capture important logical intuitions, intuitions that can be traced back to antiquity. However, the requirements that are imposed on connexive logic are actually not enough to do justice to these intuitions, as I will argue. I will suggest how these demands should be strengthened.

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A very interesting, if not well known, family of non-classical logics is subsumed under the label "connexive logics". The title, according to [8], "suggests that systems of connexive logic are motivated by some ideas about coherence or connection between premises and conclusions of valid inferences or between formulas of a certain shape". This sounds very much like the motivation behind relevant logics, but connexive logics are somewhat different beasts. Nonetheless, many researchers share a common interest in relevant and connexive systems because of the similar motivation.

The central distinguishing feature of connexive logics is that they obey the following principles:

Aristotle

$\neg(A \to \neg A)$ and $\neg(\neg A \to A)$ are valid.

Boethius

$(A \to B) \to \neg(A \to \neg B)$ and $(A \to \neg B) \to \neg(A \to B)$ are valid

These two (or rather four) principles may seem plausible, but very few modern systems of propositional logic contain them as theorems. In classical logic, for example, all of them are false if A is false. In this case, the implicational connective denotes simply the material conditional, i.e., $A \to B$ is equivalent to $\neg A \vee B$. Thus, for instance, the first form of ARISTOTLE is classically equivalent to $\neg(\neg A \vee \neg A)$, which is simply equivalent to A. However, even in non-classical logics in which the implicational connective is not the material conditional, ARISTOTLE and BOETHIUS usually fail.

Unsurprisingly, the two names for these principles derive from the ancient philosophers Aristotle and Boethius, and it is assumed that they capture important tenets that these two philosophers held. It is not quite clear how well this analysis really fits to what Aristotle and Boethius indeed had in mind, and this paper is not an exegetic endeavor to answer this question. This is because the connexive literature rarely delves into this topic and rather, after a brief tip of the hat in the direction of the ancient masters, appeals to intuitions about the validity of the schemata in question that we all are supposed to share. These intuitions are what I intend to home in on in this piece.

What are those intuitions, then? Informally expressed, they come to this:

Int1

Surely it is not the case that a proposition A should imply its own negation (or the other way around).

Int2

> Surely if A implies B, then A does not imply not-B (and if A implies not-B, then A does not imply B).

I take it that these are indeed robust pre-theoretical intuitions, and that it is at least an interesting project to try and do justice to them. However, I want to argue that the mere fact that Aristotle and Boethius are obeyed by a logic is not enough to fully answer to these intuitions.

The problem is the following: Even amongst connexive logics, there are some that allow for the satisfiability of $A \rightarrow \neg A$, and some that allow for the simultaneous satisfiability of both $(A \rightarrow B)$ and $(A \rightarrow \neg B)$.

Here are two examples of connexive systems to illustrate this point. They are not necessarily the most interesting connexive systems in the literature, but they (a) exemplify the aspects that are important for this paper, (b) are more fully discussed in the easily accessible overview article [8] and are (c) especially easy to characterize semantically. Both systems are based on many valued matrices, the first on a four valued one, the second on a three valued one.

The first system is in fact one of the earliest characterizations of a connexive logic, given by Angell ([2])[1]. Here are the tables that specify how the four values (1,2,3 and 4) are assigned to complex formulae:

[1] An axiomatization of this logic was later given in [4].

$\neg$	
1	4
2	3
3	2
4	1

$\wedge$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	3	4
4	4	3	4	3

$\rightarrow$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	1	4	1	4
4	4	1	4	1

The designated values are 1 and 2. It is easy to check that both ARISTOTLE and BOETHIUS hold in this system, but it has often been observed that it is not easy to make intuitive sense of the four truth values.

Now, for a more recent example, consider the following matrices, given by Cantwell ([3]) :

$\neg$	
T	F
F	T
-	-

$\wedge$	T	F	-
T	T	F	-
F	F	F	F
-	-	F	-

$\vee$	T	F	-
T	T	T	T
F	T	F	-
-	T	-	-

$\rightarrow$	T	F	-
T	T	F	-
F	-	-	-
-	T	F	-

Here, both T and - are treated as designated values. Again, it is easy to check that ARISTOTLE and BOETHIUS hold. However, observe that in contrast to Angell's system, here $A \rightarrow \neg A$ is satisfiable, and $(A \rightarrow B)$ and $(A \rightarrow \neg B)$ are satisfiable simultaneously (let A and B both take the value -). It turns out that this is true of other connexive systems, as well.

It seems clear to me that a logic like Cantwell's that allows for the satisfiability of $A \rightarrow \neg A$ and for the simultaneous satisfiability of both $(A \rightarrow B)$ and $(A \rightarrow \neg B)$ clearly violates the original intuitions that first prompted the adoption of ARISTOTLE and BOETHIUS. I therefore claim that, as far as connexive logicians want to make a bid to respect Int1 and Int2, the actual class of logics they should consider is more narrow than currently assumed, and needs to be delineated by a more demanding criterion. To state and label the two additional requirements I require clearly:

> **Unsat1**
>
> In no model, $A \to \neg A$ is satisfiable (for any A).
>
> **Unsat2**
>
> In no model $(A \to B)$ and $(A \to \neg B)$ are satisfiable (for any A and B).

Call a logic that satisfies Aristotle, Boethius, Unsat1 and Unsat2 a *strongly connexive logic*, and one that satisfies Aristotle and Boethius, but not Unsat1 and Unsat2 *weakly connexive.* My argument is simply that only strongly connexive logics have any claim to answer to Int1 and Int2. This is not to say that logics that are only weakly connexive, such as Cantwell's system, have no intrinsic interest, just that their interest must lie elsewhere. Indeed, Cantwell's logic is a good example: it is proposed as one that deals with what he calls "conditional negation", a topic that might be of interest completely irrespective of Int1 and Int2.[2]

Now, one might wonder whether the criterion of strong connexivity can be expressed in some manner in the object language itself, given that Aristotle and Boethius are not up to this task.

A look at an analogous problem might bring us to some interesting ideas here. Classical

[2]Similarly, the constructive ideas underlying the weakly connexive logic C introduced in [7] by themselves make it worthwhile to study this system. On the other hand, there are strongly connexive systems that have an independent motivation, such as the systems found in [5]. These systems are supposed to capture the idea that negation is essentially a *cancellation* operator, so that a contradiction basically has no content.

logic and most non-classical logics validate the principle of *explosion*, $(A \wedge \neg A) \rightarrow B$[3]. Let us remind ourselves of a philosophical argument that is sometimes given in favor of the validity of $(A \wedge \neg A) \rightarrow B$. This validity, it is claimed, is the most we can do to express, in the object language itself, the thought that a contradiction is unsatisfiable.

In analogy to this use of explosion to express the unsatisfiability of any contradiction, we might try to ask that $(A \rightarrow \neg A) \rightarrow B$ should be valid, in order to express in the object language that $A \rightarrow \neg A$ is unsatisfiable (and similarly for the rest of the connexive theses). Call a logic that validates all of these schemata and satisfies all the requirements for strong connexivity *superconnexive*. This criterion would delimit an even more exclusive family of logics than strong connexivity does: The Angell/McCall system, even though it is strongly connexive, would be ruled out.

Another way of arguing for the suitability of superconnexive systems is by the following consideration: Given a classical metatheory and the definition of logical consequence as "Whenever the premisses take a designated value, the conclusion does, too", the inference $(A \rightarrow \neg A) \vDash B$ will hold in a strongly connexive system. In those cases in which $(A \rightarrow \neg A) \rightarrow B$ is not valid, the deduction theorem would fail, something that (arguably) we would like to hold.

On the other hand, one might well have doubts concerning superconnexivity. A superconnexive logic requires an implication that always turns out to be true if the antecedent is unsatisfiable; many implicational connectives in the literature indeed

[3] Any logic that does *not* validate explosion is called a *paraconsistent* logic. Relevant logics are typically paraconsistent, as are logics that are constructed to deal with paradoxes, if these are conceived of as true contradictions (cf. [6]).

Incidentally, all non-trivial systems that are only weakly connexive must be paraconsistent, for if $A \rightarrow \neg A$ is satisfiable and $\neg(A \rightarrow \neg A)$ is a logical truth, then these contradictory statements may not imply any statement what so ever. So, $(A \wedge \neg A) \rightarrow B$ cannot be a logical truth (unless one is willing to give up modus ponens).

behave this way, but there have also been deep reservations about those conditionals voiced by relevant and paraconsistent logicians. In the case at hand, they would balk at the validity of $(A \to \neg A) \to B$ because there is no connection between the antecedent and the consequent at all. Likewise, $(A \to \neg A) \vDash B$ would not meet what they would require of a decent consequence relation. Given the similarity of the motivations behind the connexive and the relevant project mentioned in the first paragraph, superconnexivity might be ideologically suspect.

More importantly, embracing superconnexivity means to say good-bye to substitutivity[4]. For if we are allowed to substitute $\neg(A \to \neg A)$ for B in $(A \to \neg A) \to B$, we get $(A \to \neg A) \to \neg(A \to \neg A)$, which is against the requirement of strong connexivity. This seems like a drawback that will deter many, even though some connexive logics dispense with substitutivity anyway, such as the logics introduced in [5].

In any case, I believe that a weakly connexive logic can not claim to be the solution to "the *idee maitresse* of connexive logic that no proposition can be incompatible with itself, and hence cannot entail, or be entailed by, its own negation"[5]. At least strong connexivity is called for here.

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[4]David Makinson pointed this out to me after a quick glance at the proposal.
[5][1], p. 437.

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