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From Semantics to Logic: The Scenic Route

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RESUMEN

Este artículo reconsidera un hito importante del argumento de Michael Dummett a favor de la lógica intuicionista: el paso de una teoría del significado a una explicación de la consecuencia lógica. Ofrece además una orientación para aquellos que no están tan familiarizados con la semántica para la lógica intuicionista como para seguir aquí a Dummett fácilmente. Al mismo tiempo, señala importantes cuestiones que él dejó todavía abiertas y sugiere cómo una combinación de la interpretación BHK y la semántica de Kripke para la lógica intuicionista podría resultar que fuera lo mejor para sus propósitos.

PALABRAS CLAVE: *lógica intuicionista, semántica para las lógicas no clásicas, interpretación de Brouwer-Heyting-Kolmogorov, semántica de Kripke*.

ABSTRACT

This essay revisits an important step in Michael Dummett's argument for intuitionistic logic: The move from a theory of meaning to an account of logical consequence. It offers an orientation for those who are not as acquainted with the semantics for intuitionstic logic to easily follow Dummett here. At the same time, it points out important questions that are still left open by him, and suggests how a combination of the BHK interpretation and the Kripke semantics for intuitionistic logic might turn out to best serve his aims.

KEYWORDS: *Intuitionistic Logic, Semantics for Non-Classical Logics, Brouwer-Heyting-Kolmogorov Interpretation, Kripke Semantics.*

I. INTRODUCTION

One of the central ideas of Michael Dummett's work is this: We can find interesting answers to metaphysical problems by inspecting the validity or failure of certain logical laws, chief among them the law of excluded middle. Whether these laws hold will be decided by attending closely to the meaning of the logical constants. This can only be done with a firm understanding of meaning in general, so that the philosophical journey will have to start with the analysis of language.

Part of Dummett's legacy is, I believe, to retrace and rearrange his arguments in various ways to get clearer on the lay of the land he leads us through. In this essay, I want to focus on how we can move from a theory of meaning to an account of logical consequence. This is a very important step in his argumentation. It is also a step that can be confusing to philosophers who have had only limited training in the semantics of non-classical logic, but also to logicians who are not used to think as exhaustively (and sometimes exhaustingly) about the nature and purpose of semantic theories as Dummett did. To make it more easy to follow, I will keep it as simple as I can (by, for example, confining myself to propositional logic), refrain from turning into many tempting philosophical byroads and explain the remaining ideas with more detail than Dummett at times had the patience for. In turn, by going more slowly, some important philosophical questions come up that are not perceived if one moves too swiftly.

II. Semantic Theories

To get started, we need to clear up some Dummettian terminology that could otherwise generate confusion:

- A MEANING THEORY for a particular language records all that a speaker needs to know, whether explicitly or implicitly, in order to be considered a competent speaker of that language.

- A THEORY OF MEANING, in contrast, gives the general form in which a meaning theory has to be presented. "The task of a theory of meaning is to give an account of how language functions, in other words, to explain what, in general, is effected by the utterance of a sentence in the presence of hearers who know the language to which it belongs" [Dummett (1991), p. 21].[1] Here, statements that are asserted (as opposed to questions, commands, etc.) take theoretical priority; Dummett thought that all other types of speech acts are derivative of the act of assertion.

- A SEMANTIC THEORY is a theory of how the correct assertibility (or the truth) of a statement is determined by its semantic value, and how its semantic value depends on the semantic values of its parts. Whether correct assertibility or truth should be the central notion is something that Dummett seems to have changed his mind about more than once. In both cases, though, the hinge to the theory of meaning is the fact that the semantic theory tells us in which cases it is correct to utter a statement in the assertoric mood.

Because it is concerned with semantic values, the semantic theory will have to answer questions about the reference of singular terms and the like. As we are interested in questions of logic (and are going to concentrate on the propositional case), the crucial question is this: How do logically complex statements receive their semantic values, and in what way do they depend on their constituent statements? For example, if we know the value of *A* and the value of *B*, how do we get to the value of "*A* and *B*", "If *A*, then *B*" and so on?[2] A semantic theory that only gives us this information, without telling us much about the atomic statements, is called a "skeletal" semantic theory by Dummett. Though not satisfying on the whole, such a skeletal theory will give us enough information to determine what follows logically from what, as we will see below.

The nature of the semantic values of statements will decide whether our theory is a *constructive* one or not. If we choose semantic values that are epistemically accessible (e.g. proof conditions, verification conditions, falsification conditions), then we are setting up a constructive semantics. A non-constructive semantics will have values that might be well beyond our epistemic grasp. Dummett thinks those theories are ruled out by very fundamental considerations about language.

Dummett's prime example of a constructive semantic theory is the account of mathematical statements in Brouwer's intuitionism: Such a statement may be asserted only if it is provable. The meanings of the logically complex (i.e., non-atomic) statements are in turn given in terms of proofs of the constituent statements. We will see how this works in detail below.

Once we have found out the semantic value of a statement, the next task of the semantic theory is to tell us whether the statement is correctly assertible/true in a given situation or not. With this information, we can finally move on to a conception of logical consequence: An inference will be valid if it never fails to transmit correct assertibility/truth from premises to conclusions.

Let's see by way of example how the non-constructive semantic theory that gives the meaning of the classical logical constants works. It is as well known as it is simple. There is no mystery in how the semantic value of a statement relates to truth or falsity: The semantic value of a statement simply *is* its truth value.

The determination of the truth value of a complex statement and the account of logical consequence is completely taken care of by the truth tables of classical propositional logic and, crucially, the assumption of bivalence:

Bivalence: Every statement is determinately[3] either true or false.

However, constructivists have a problem with this principle, which lies in the existence of undecidable statements. An undecidable statement is one, such as Goldbach's conjecture, that we have at present no idea how to solve. In dealing with an undecidable statement, we have nothing that guarantees that we can come to know of its being true or false. But lacking such a guarantee, a constructivist does not feel entitled to claim that the statement is either true or false.

To give up bivalence means already to give up an important part of classical logic. There *might* be an alternative story that does not invoke bivalence and still motivates all inferences of classical logic.[4] It is more natural to assume, however, that without bivalence, some classical inferences will lose their validity.

This is exactly what happens in intuitionistic logic, which is well known for the inferences it doesn't validate. The first of these is the failure of the Law of Excluded Middle (LEM):

LEM $\vDash A \vee \sim A$[5]

Even those unacquainted with intuitionistic doctrine will have no trouble seeing why this law must be dubious if bivalence cannot be assumed. Such lucid plausibility can hardly be claimed for the second most famous characteristic feature of intuitionistic logic, the failure of Double Negation Elimination (DNE):

DNE $\sim\sim A \vDash A$

To understand this failure, one needs to understand the way in which negation is explained in the semantics for intuitionistic logic, which will be explained presently.

III. Various Semantics for Intuitionistic Logic

Over the years, there have been a number of semantical theories proposed for intuitionistic logic, some of which are [cf. Dummett (2000)]:

- The so called Brouwer-Heyting-Kolmogorov interpretation (BHK for short), which spells out the meanings of the logical constants directly in terms of proofs.
- The topological interpretation which interprets intuitionistic formulas as open sets in a topology
- Beth trees, in which different stages of mathematical investigation are related to each other, quite similar to the next item

- Kripke semantics, which I will introduce in detail below
- Kleene's realizability semantics, where formulas are interpreted as codes for algorithms
- Dialogue semantics, where intuitionistic validity comes to having a winning strategy in a dialogue game

These interpretations are not the only ones around, but they already display the variety of guises a semantic theory might take, and their quite different strengths and weaknesses. An important spectrum along which they differ is how simple they make meta-theoretical investigation vs. how much they add to our intuitive understanding of the formulas.

At the respective extremes of this spectrum we find the topological interpretation and the BHK interpretation. The topological interpretation tells us how we can translate our knowledge about topology to gain insights into the formal properties of intuitionistic logic; at the time it was found, this meant a considerable information gain, because the topology of open sets was widely studied, while intuitionistic logic was a new topic of investigation. However, it is impossible to glean anything of the philosophical motivation from the association with open sets,[6] and Dummett can't see the topological interpretation delivering all he is hoping for in a semantics:

> In attempting to find a semantics with respect to which a given logical system is both sound and complete, a logician is not merely seeking an algebraic rather than a proof-theoretic characterization of the deducibility-relation of that system: a semantical theory proper (...) is to be distinguished from a merely algebraic valuation system (such as the topological interpretation (...)). If any distinction between a semantical theory and a purely algebraic valuation system can be drawn at all, the ground of it must lie in the fact that the semantical theory is connected with the way in which both logical and non-logical expressions are given meaning, while a purely algebraic one is not [Dummett (2000), p. 256; cf. also Dummett (1978), p. 293, and Dummett (1998), p. 126].

IV. The BHK Interpretation

In contrast to the topological interpretation, the BHK interpretation does give us the meaning of the logical constants and is thus a candidate for a semantic theory in Dummett's sense. These meanings are spelled out in terms of *proof conditions*.

The BHK interpretation gives us a very good idea of what it is to prove a complex statement, provided we know what it is to prove statements of lesser complexity. To get a basic idea of how this will work, consider a conjunction of two statements, the proof conditions of which we assume to know. That is, we could come to recognize a proof of either of them if we were presented with one. What, now, would a proof of the conjunction look like? Well, it would simply be a proof of the first conjunct, followed by a proof of the second conjunct.

A proof of a disjunction under BHK is a proof of either disjunct, together with an indication which disjunct is proved.

What about negations? The intuitionist tries to express everything in terms of constructions and proofs that are obtainable by us. A negation, in contrast, seems to be telling us something about the *impossibility* of proof and construction. How might this be expressed in constructivist terms?

Consider how we go about proving negative statements in mathematics, such as the claim that there is no greatest prime number. The first step is to assume that there is a greatest prime number. Then we reason from that assumption, until we hit a contradiction. In the case of Euclid's famous proof, we prove that there is a greater prime number than the one we assumed to be the greatest, which obviously contradicts the assumption.

The proof of a negation is thus a fairly elaborate process, and it seems at heart to be a kind of conditional. *If* there is a greatest prime number x, *then* there is another one that is even greater than x. Or, to make the contradiction perfectly explicit: *If* there is a greatest prime number x, *then* x is the greatest prime number and x is not the greatest prime number.

This leaves us with two questions: How does the constructivist explain *conditionals*? We'll see that below by directly considering the BHK clause, but before that, let us consider the second question: Isn't our definition of negation circular? We are trying to explain what negation means by taking recourse to a conditional that has a contradiction as its consequent. But a contradiction is obviously a notion that presupposes the notion of negation.

The intuitionists employ the following trick: Instead of a contradiction, they use a mathematical statement that can have no proof, usually $1 = 0$. This absurd statement is abbreviated as $\bot$. Mathematics is conceived as endowed with such an inherent coherence that the assumption of *any* false statement will eventually lead to a proof of $1 = 0$.

Here is one version of the clauses for the sentential connectives:

- c is a proof of $A \wedge B$ iff c is a pair $(c1, c2)$ such that $c1$ is a proof of A and $c2$ is a proof of B.
- c is a proof of $A \vee B$ iff c is a pair $(i, c1)$ such that $i = 0$ and $c1$ is a proof of A or $i = 1$ and $c1$ is a proof of B.

- c is a proof of $A \supset B$ iff c is a construction that converts each proof d of A into a proof $c(d)$ of B.

- nothing is a proof of $\bot$.

- c is a proof of $\sim A$ iff c is a construction which transforms each proof d of A into a proof $c(d)$ of $\bot$.

We now see how conditionals are dealt with. A conditional is taken to be a construction that delivers a proof of the consequent if supplied with a proof of the antecedent. We also see what was noted above, that the clause for negation has the same structure as the clause for the conditional.

Indeed, often enough, the last clause is left out, because negation is simply *defined* as an implication of absurdity: $\sim A$ is short for $A \supset \bot$. Again, the idea that is supposed to be captured is that, when you are able to derive an absurdity from a hypothesis, you can be sure that you will never come across a proof for that hypothesis.

The above clauses do not tell us what kinds of objects proofs are, nor how proofs for atomic statements might be structured. This is why it only provides a skeletal semantics. However, we can already see that pairs of proofs such as the ones that make up the proofs of conjunctions and disjunctions, and constructions that transform proofs into other proofs, such as the proofs for negations and conditionals, count as proofs. From the general setting in the constructivists' argument, we can also glean that proofs are things that are epistemically accessible.

It is the explanation of complex statements' semantic values as something inherently recognizable (proof conditions) that makes the intuitionistic account so attractive to Dummett:

> The fundamental idea [of intuitionistic semantics] is that a grasp of the meaning of a mathematical statement consists, not in a knowledge of what has to be the case, independently of our means of knowing whether it is so, for the statement to be true, but in an ability to recognize, for any mathematical construction, whether or not it constitutes a proof of the statement (...). The understanding of any mathematical expression consists in a knowledge of the way in which it contributes to determining what is to count as a proof of any statement in which it occurs. In this way, a grasp of the meaning of a mathematical sentence or expression is guaranteed to be something which is fully displayed in a mastery of the use of mathematical language, for it is directly connected with that practice [Dummett (1993), p. 70].

The BHK semantics offers only a skeletal semantic theory, but again, a skeletal semantics is in principle enough to specify logical consequence. To say that a statement follows from a given set of premises is to say that, if the premises are provable, the consequence is provable as well.

The LEM would, under the BHK-interpretation, be saying that it is possible to give an intuitionistic proof for every given statement or its negation. But we actually have no guarantee for such a claim. The only such guarantee that an intuitionist would accept would be a proof (or at least a secure method of obtaining a proof) of the statement in question or of its negation.

That the LEM should appear dubious to a constructionist was clear even before we saw this concrete interpretation. More interestingly, under the BHK interpretation, we can now also begin to understand why Double Negation Elimination should fail. Replacing the negations by implications of absurdity, the law can be formally stated as: $((A \supset \bot) \supset \bot) \supset A$. This would, were it valid, tell us the following (which I try to make easier to parse by employing italics, emphasis and line breaks):

> If we have a CONSTRUCTION THAT TURNS
>
> *a further construction, which turns a proof of A into a proof of* $\bot$
>
> INTO A PROOF OF $\bot$,
>
> then we could always transform this whole thing into a proof of *A*.

In other words, if we can be sure that we will never find a proof for the claim that we can be sure never to find a proof of *A*, then this is as good as having a proof of *A*.[7] But the intuitionist holds that it is not: It serves only to show that $\sim A$ will never be proven. We might well know that, without yet having a guarantee that we will find an actual constructive proof for *A*.

IV.1 *Correctness and* Ex Contradictione Quodlibet

Thus, we can deduce some characteristic features of intuitionistic logic from the BHK-interpretation. In general, however, it is doubtful whether it is concrete enough to support correctness (completeness and soundness) proofs for intuitionistic logic.

Heyting and Kolmogorov had in fact both independently proposed BHK-style interpretations. However, they were not at all in agreement whether intuitionistic logic is sound with respect to their interpretation. While Heyting did indeed think so, Kolmogorov was worried about the admissibility of the intuitionistically valid inference from a contradiction to an arbitrary statement, *ex contradictione quodlibet* ($A \wedge \sim A \vDash B$) [cf. van Dalen (2004)]. The question is whether there is a general method of generating a proof of any statement whatsoever from a proof of a statement and a proof of its negation. As $\sim A$ can be seen as the conditional $A \supset \bot$, *A* and $\sim A$ together let us infer, via modus ponens, $\bot$. To argue for ECQ, we have to give an argument that $\bot$ entails every other statement.

There are two conceivable ways for the intuitionist to argue for ECQ.

The first way of doing this is to say that it will indeed be possible to find a way to transform a given proof of 1 = 0 into a proof of any other statement. This is clearly constructive in spirit, if indeed it can be pulled off. However, it is not quite clear that it is a plausible assumption if we consider all of mathematics, not just elementary arithmetic or some other restricted area. An additional worry is that we seem here to be transforming proofs that can not exist [cf. Wansing (1993), p. 21]; it isn't perfectly clear how we can claim to be able to do that.

The second way is to say that it is quite all right to claim to have a construction that will effect the conversion of a proof of the premise into a proof of the conclusion, simply because there will be no input that would put our claim to the test.

Here, the problem is that it is not wholly clear that the argument is all that constructive. It plays much more on the impossibility of constructions than on their possibility. We claim to have a construction where in fact we don't have one, or one that ostensibly would not do the required job if, counterfactually, it were supplied with a suitable input.

Kolmogorov and others after him thought that there is something fishy about both ways this argument could be spelled out; for example, worries about this argument have led to the development of *minimal logic* [cf. Johansson (1936)], where a contradiction does not entail everything. The notions minimal logicians want to capture are not distinct from the ideas encapsulated in the BHK-interpretation, but clearly they propose a different logic.

We are then looking for something more to the center of the spread between the formal extreme of the topological interpretation and the informal BHK interpretation. What I will be concentrating on from now on are the Kripke semantics for intuitionistic logic, which strike a good balance between formality and intuitiveness.

V. KRIPKE SEMANTICS

The semantics is a variation of the well known possible worlds semantics for modal logics.[8] The worlds are supposed to be information states that the inquisitive subject steps through in her quest for knowledge. As intuitionistic logic was conceived of as a logic for constructive mathematics, the worlds are normally taken to record what has been proven up to a certain point in time. There is an accessibility relation on the worlds that has a strong temporal flavor: One stage comes literally *after* the other. Let us spell this out a bit more formally:

A model will be a structure, $[W, \leq, v]$, where W is a non-empty set of worlds or information states, and $\leq$ is a binary relation on those worlds which is reflexive, transitive and anti-symmetric, that is, a partial order.

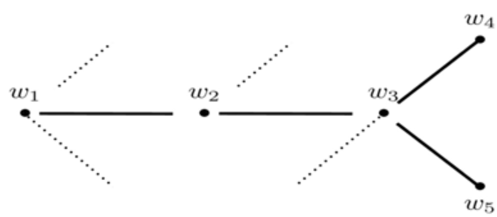
The valuation function v assigns a truth value, 1 or 0, to each atomic statement p at each world. Intuitively, for p to receive value 1 at a world means that p is proven at that world. For it to receive value 0, on the other hand, means just that p has not been proven (yet) at world w. It does not mean that p has been disproven at world w. That is, if the investigation is carried on, we might well find a proof of p at a later stage or world, and thus the value at that stage will turn to 1. On the other hand, a proof is something definitive, such that a statement that is proven at a world will always remain proven at subsequent stages.[9]

In the semantics, this thought is manifested in the so called heredity constraint:

For each p: if $w \leq w'$ and $v_w(p) = 1$ then $v_{w'}(p) = 1$.

For $v_w(p) = 1$ I will write $w \Vdash_1 p$, and for $v_w(p) = 0$ I will write $w \Vdash_0 p$, leaving out reference to v to reduce clutter.

One can represent the models graphically, which makes understanding what is going on much easier. Here is an example:



	w_1	w_2	w_3	w_4	w_5
p	0	1	1	1	1
q	0	0	0	1	1
r	1	1	1	1	1
s	0	0	0	1	0

This shows part of a model for intuitionistic logic. The lines represent the accessibility relation between the worlds, represented as dots and labeled w_n. It is meant to be read from left to right, such that, for example, w_1 should be seen as accessing w_2, but not vice versa. Because the relation is by definition reflexive, every world can access itself, a fact that is not represented in the diagram.

The dotted lines indicate other branches in the model that we choose not to consider. Note especially the dotted line leading to w_3: These models need not be trees, and a world may have more than one line leading up to it.[10] Remember that the worlds are supposed to record information states; the idea that lies behind allowing worlds with more than one such lines is that one may come to the same state of information via different routes. Say we will from now on turn to prove two mathematical statement, say, the Goldbach conjecture and the P = NP problem. One way to do this is to first turn to Goldbach, and then take care of P = NP, another way is to do it the other way around. If we solve nothing but those two problems, we should come out at the same state of information, no matter in which order we tackle them.

Now, take a look at the valuation below the diagram. In this matrix, the fate of four propositional constants is recorded. The workings of the heredity constraint come out clearly here: Once a proposition is proven, that is, once it receives value 1, it stays proven. At the first world, the only thing we have proven is the proposition r, which will never go back to being unproven (receiving value 0), no matter what else happens. At w_2, we manage to prove p, and so we have two atomic propositions that get assigned value 1.

Passing from w_2 to w_3 seems to bring little change. There might be other statements which we haven't listed, but maybe there aren't. There is nothing precluding two or more successive worlds to be indistinguishable in terms of the statements that are proven at them. If we want to think of the worlds as marking points in time, then in such a stretch of indistinguishable worlds, there is simply no progress made in our inquiry.

In contrast, we register an impressive increase in our knowledge when we move from w_3 to w_4: Both of the remaining unproven atomic statements, q and s, are proven simultaneously. Again, there is nothing in our definition of intuitionistic models preventing this.

The transition from w_3 to w_5 marks a different possible way our investigations may unfold: Only one of the statements is proven, while the other stays open.

V.1. *Logical Constants*

So much for atomic statements. Next, here are the conditions that determine the semantic values of logically complex statements:

For all $w \in W$:

$w \Vdash_1 A \wedge B$ iff $w \Vdash_1 A$ and $w \Vdash_1 B$
$w \Vdash_1 A \vee B$ iff $w \Vdash_1 A$ or $w \Vdash_1 B$

$w \Vdash_1 A \supset B$ iff for all $x \geq w$, $\Vdash_0 A$ or $x \Vdash_1 B$
$w \Vdash_1 \sim A$ iff for all $x \geq w$, $x \Vdash_0 A$

Under these conditions, the heredity constraint holds for all statements, not just the atomic ones.

The success of the semantics lies in the fact that most people see these clauses as a formal precisification of the BHK-interpretation, and not really a rival semantic proposal. Let us see in how far the two accounts of complex statements are in harmony.

To repeat, the inquiring subject knows a statement to be provable at a given world if she has a proof of it. If that statement is a conjunction, then that proof consists of a pair of proofs, one for each conjunct. Therefore, she has these proofs at her disposal, and thus the two conjuncts are proven on their own. That is, at the world in question, each of the conjuncts will get the value 1 assigned to it, just as the clause above tells us.

Disjunctions are just as straightforward. If our subject has a proof of a disjunction, then BHK tells us that she has something that amounts to a proof of either disjunct. Therefore, either the first or the second disjunct will get the value 1 at the world we are concerned with.

It gets interesting when we look at the intensional notions, that is, those that look to other worlds in order to determine the truth-value of a statement at the present world. A glance back at the clauses tells us that this concerns conditionals and negations.

First of all, the intensional notions make clear that knowing whether a statement receives value 1 or value 0 at a world is not enough to know the semantic value of the statement, in the sense of Dummett. The semantic value is not just telling us whether a statement is assertible or not, but also how a complex statement in which the statement is a constituent should be evaluated. For this, we will need to know the fate of the statement in the accessible worlds, and that must then be seen as part of the semantic value as well. It is easy to overlook this, due in part to somewhat unfortunate terminology, but also due to the fact that one tends to focus on the value 1 too much. Given the heredity constraint, it makes no difference whether we say that the semantic value of a statement that receives value 1 is exhausted by that, or whether we say that we need to know the future development of that statement, for in all accessible worlds it will also receive value 1. If the value is 0, on the other hand, we can not say whether, for example, the negation of the statement receives value 1 or value 0. This will depend on the question whether the original statement will receive value 0 in all subsequent worlds or not.[11]

With that in mind, let us look at how exactly these intensional notions work. A proof of a conditional, BHK tells us, is a construction that takes any proof of the antecedent and turns it into a proof of the consequent. That is, if a conditional is true at a world, it is inconceivable that the inquiring subject should acquire a proof of the antecedent and not be able to prove the consequent. Therefore, we see that in all possible developments the investigation might take from the point in which the conditional is proved, a proof of the antecedent will always immediately result in a proof of the consequent: In all worlds that represent possible developments, either the antecedent is unproven (has value 0) or the consequent is proven (has value 1).

A negation is proven, the BHK interpretation explained, if we have a construction that will turn any proof of the negated statement into a proof of an absurdity. Such an absurdity, of course, will never be proven, and thus there will be no world at which it will receive value 1. Therefore, no world that is a possible development of the present stage of investigation will assign value 1 to the negated statement, or we would be forced to assign 1 to the absurdity.

This tells us something about the kind of modality that is involved in the Kripke interpretation. The possible developments the interpretation talks about can only be epistemically possible, not mathematically: If they had to be mathematically possible, then no world would assign value 1 to a false mathematical statement, and thus the negations of all false statements would be true from the outset.

V.2. *Logical Consequence*

Once a semantic theory has told us about the semantic values of atomic statements and how to compute the values of complex statements on that basis, it will have to tell us how these values relate to truth. For one thing, because we are interested in truth per se, but more importantly because we want to define logical consequence in terms of truth preservation.

The semantic values in our Kripke semantics are 1s and 0s. I said that a statement receives value 1 at a stage or world iff it is proven at that stage. Of course, once this happens it is clear that the statement is true. However, how about those stages that came before, where the now proven proposition had not yet been proven? Clearly, the semantics tells us that it will have received value 0 at these stages. But do we want to say that it had not been true at those earlier stages, even though we went on to prove it?

This is a very tough question for the constructivist, but unfortunately I don't have space enough to go into it here. We'll simply have to assume that in our models, only those statements are true that receive value 1, and that this value is what the consequence relation is to transmit:

$\Gamma \vDash A$ iff in every model and every $w \in W$, if $w \vDash_1 B$ for every $B \in \Gamma$, then $w \vDash_1 A$.

With consequence thus defined, the Kripke semantics is sound and complete with respect to Heyting's axiomatization.

It is not hard to come up with counter models for the characteristic failures of classical validities in intuitionistic logic:

Let $w_1 \Vdash_0 p$ and $w_2 \Vdash_1 p$. This is a counter model for both LEM and DNE.

LEM: We have that $w_1 \Vdash_0 {\sim}p$ because p does not receive value 0 at all subsequent worlds. Therefore, at w_1, $p \vee {\sim}\, p$ receives value 0.

DNE: We obviously also have that $w_2 \Vdash_0 {\sim}p$. Thus, ${\sim}p$ receives value 0 at all worlds accessible from w_1. But that means that $w_1 \Vdash_1 {\sim\sim}p$, and from that we can compute that $w_1 \Vdash_0 {\sim\sim}p \supset p$.

VI. Joining Semantic Theories

If we want to give more concrete content to the BHK interpretation, then the Kripke semantics works rather well; at least it gives one way of answering the questions that were not quite clear about BHK, for example whether *ex contradictione quodlibet* should hold or not. But do we even need the BHK interpretation, or would the Kripke semantics alone suffice?

The answer seems to be that we have a mutual dependence here: The Kripke semantics will need the heuristics of the BHK interpretation (or something like it) to put it in a better position than the topological interpretation.

For example, consider the clause for negation: A negation is true iff the statement that is negated is unproven in all accessible worlds. But what if we are in a model in which it just happens, for no particular reason at all, that in all accessible worlds from the actual world, there is a statement that is unproven throughout? The negation of that statement will be true, but why? Isn't this just chance?

Similar questions arise about the conditional. In a model in which there just happens to be no world in which the antecedent is proven and the consequent isn't, the corresponding conditional will be true throughout. But it could be quite a coincidence that matters stand as they do in the model. This would seem somewhat unsatisfying, just as the classical, material conditional is unsatisfying, as it holds between pairs of statements that might be quite unrelated to each other, as long as the first is true and the second false.

To begin to explain we should again mention, as I did above, that the alternative worlds have to be seen as epistemic alternatives. To make sense of negations and conditionals, we need to provide accessible worlds for all the epistemic alternatives that we can conceive of.

But why *is* there no proof of *p* in the subsequent worlds? What makes us look at such a model, and not one in which there is a later proof of *p*? The answer is that the fact that there will be no such world is already contained in the information available at the actual world. There is something that we know that tells us that *p* will not be proven.

It seems that the models themselves, or at least what we have said about them until now, do not give us a good answer to the question what exactly it is that we know here. To give such an answer, we have to supply some additional explanation. If this explanation is to be true to the constructive spirit of intuitionistic logic, we presumably have to go back to an account in terms of proofs. Such an account is, in presentations of this semantics, often given in a more or less hand waving manner. However, we have already seen a relatively elaborate account that we can utilize, namely the BHK-interpretation.

Indeed, the BHK clause for negation gives us quite a good indication of how to model a state of information in which a negation is proven. It is the fact that we have a means of turning a proof of *p* into a proof of $\perp$ that lets us choose a world in a model in which no successor world proves *p* as a suitable representative of such a state of information. Likewise, the models in which a conditional is proven at a world look the way they do (i.e., the worlds the world can access will either assign value 0 to the antecedent or value 1 to the consequent) because we know of a method of transforming a proof of the antecedent into a proof of the consequent.

It seems then that we really need a BHK-style interpretation to supplement a Kripke semantics to give meaning to the logical constants of intuitionistic logic.[12] On the other hand, the Kripke semantics supply enough mathematical detail to give soundness and completeness proofs. For example, it will be clear from the clauses for the connectives and the definition of validity that *ex contradictione quodlibet* is valid under the BHK interpretation, as precisified by the Kripke semantics.

In sum, the Kripke semantics for intuitionistic logic and the BHK-interpretation do seem to make a good couple. Together, they explain how complex statements receive their semantic values from the values of their constituents, and they connect these semantic values to something epistemically accessible, namely proof conditions. By identifying truth and provability, we get a clear answer to the question of logical consequence.

VII. Further Along the Way

I will leave you here by making the obvious observation that there is much more to be explored ahead:

There is the vexing issue whether truth might really be a tensed notion in the way the semantics suggests by identifying truth with provability. Can a statement of mathematics literally become true, just because someone found a proof of it?

Related to this point, we should address the question whether we are dealing with explicit or implicit knowledge in stating our semantic values, a distinction that hides in the words “proven” and “provable” and in their inherent ambiguity. Likewise, the notion of an undecidable statement is not at all as clear as I have made it sound.

Once these points have been cleared up, there is the next step that needs to be taken: Until now, we have only been giving a semantics for mathematical statements. Dummett, however, wants to take this as a blueprint to account for the semantics of empirical language as well.

On the face of it, this seems to just involve a substitution of verification conditions for proof conditions. However, worries he himself has voiced suggest that the account can not be transplanted that easily, and following up on these worries might lead us to quite different non-classical logics than intuitionistic logic.

For example, he at times entertained the idea that in empirical discourse, the correctness of an assertion should not depend on its verifiability, but on its unfalsifiability [cfr. Dummett (1993), p. 83]. This change in the theory of meaning would have a profound effect on the semantic theory: If we still wanted to identify truth and verifiability, then truth could no longer be the central notion of the semantic theory. Truth would not be what is transmitted by logical consequence; rather, it should transmit non-falsitiy.[13] It is in ways like these that the semantic theory depends on basic considerations in the theory of meaning, acquiring features that will be passed on to the specific meaning-theories for specific languages.

And lastly of course, way over yonder in the distance, there are the metaphysical conclusions we may or may not come to draw from all this.[14]

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NOTES

[1] Maybe “theory of language use” would have been a better label. Not only easier to discern from “meaning theory”, it would have made the rest of this quote sound less puzzling: “The notion of meaning itself need not, therefore, play an important

role in a theory of meaning" (ibid). It might not, if there were no tight connection between meaning and use. However, Dummett thinks such a tight connection holds.

[2]Semantic theories that do not presuppose compositionality will have to explain what else is needed, over and above the values of *A* and *B*. Dummett does not, however, consider such theories [cf. Dummett (1991), p. 25]. Note also that in order to give these rules and in order to fix logical consequence, semantic theories deal with schematic sentence letters and thus with a notion of truth under an interpretation, not just truth simpliciter, cf. Dummett (1991), pp. 20.

[3] On the force of this qualification, cf. Dummett (1991), p. 75.

[4] Ian Rumfitt has suggested such a strategy in Rumfitt (2007). Another way to achieve this is supervaluationism.

[5] $\models$ stands for the semantic consequence relation.

[6] The referee did not think this a fair comment and rightly pointed out that the Kripke semantics I will introduce below can mathematically be seen as a topological semantics. However, as will become clear, it is the additional layer of interpretation that the Kripke semantics supplies that puts it ahead of a 'merely' topological semantics that lacks such a layer. In talking about topological semantics, I (and Dummett) mean only the latter.

[7] This gloss in epistemic and temporal terms foreshadows the intuitive interpretation that we will find in the Kripke semantics below.

[8] Dummett actually prefers the closely related Beth semantics; however, except for the treatment of disjunctions, the differences are not too great, and the Kripke semantics are easier to understand for various reasons.

[9] Note also that the fact that the valuation function is a function suggests that it is a decidable question whether or not a given statement has been proven at a given stage. This does not mean that the question whether a statement is in principle provable or unprovable is decidable, though.

[10] As far as intuitionistic logic is concerned, it does not hurt to restrict one's attention to trees. The consequence relation that arises from tree models is no different from the consequence relation one gets from the more inclusive set of partially ordered models.

[11] The referee points out that it is a striking fact that a statement and its negation may both be true in the same model, albeit at different worlds. To keep this in harmony with the BHK interpretation, we must remember that our present state of information is represented by a single world, not a collection of them.

[12] Dummett comes to a similar conclusion, although he is concentrating in his discussion on Beth trees [Dummett (2000), p. 287].

[13] It should also be mentioned that there is yet another, altogether different approach Dummett has suggested to account for the meaning of logical vocabulary which is known as proof-theoretic semantics, where inference rules and not semantic values determine logical consequence.

[14] I would like to thank Hannes Leitgeb, Dave Ripley, Sven Rosenkranz and Heinrich Wansing for comments on an earlier vesion of this material. Additionally, a referee for **teorema** made a large number of astute observations that have helped to improve the piece. The work has been supported by the Spanish Ministery for Science and Innovation, the Alexander von Humboldt Foundation and the German Research Foundation (DFG).

REFERENCES

DUMMETT, M. (1978), *Truth and Other Enigmas*, Harvard, Harvard University Press.
— (1991), *The Logical Basis of Metaphysics*, London, Duckworth.
— (1993), *The Seas of Language*, Oxford, Clarendon Press.
— (1998), 'Truth From the Constructive Standpoint', *Theoria*, vol. 64 (2-3). pp. 122-138.
— (2000) *Elements of Intuitionism*, Oxford, Oxford University Press.
JOHANSSON, I. (1936) 'Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus', *Compositio Mathematica*, vol. 4, pp. 119-136.
RUMFITT, I. (2007), 'Asserting and Excluding: Steps Towards an Anti-Realist Account of Classical Logic', in Auxier, R. (ed.), *The Philosophy of Michael Dummett*. Peru, Illinois, Open Court Publishing Company, pp.1059–1063.
VAN DALEN, D. (2004), 'Kolmogorov and Brouwer On Constructive Implication and the *Ex Falso* Rule', *Russian Mathematical Surveys*, vol. 59, pp. 247–257.
WANSING, H. (1993) *The Logic of Information Structures*, Berlin, Springer.